Double integrals

1) $$\int_0^1 \int_0^1 (x+y)^2 \, dx \, dy$$

Simple iterated integral.

2) Integrate x tan y over the region

$$T = \{ (x,y) : \underbrace{x+y \leq 1, \ x \geq 0 \ , \ y \geq 0} \}$$

We will do it both ways. First over x and then over y and vice-versa.

1) $$\int_0^1 \int_0^1 (x+y)^2 \, dx \, dy$$

Simple iterated integral.

$$\iint_R x \tan y \, dx \, dy$$

First Steps

1. Plot the region
2. Can you integrate x or $\tan y$?

Change of variable.

Integrate $f(x,y) = xy$ over the domain

$\{ (x,y) : x^2 + y^2 \leq 2 \} = S$

6.2 Jointly Continuous Distributions

Will be dealing in general with an object called

the JOINT PROBABILITY DENSITY FUNCTION

or joint pdf. Given two rvs X, Y and a function

$f_{X,Y}(s,t) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ such that

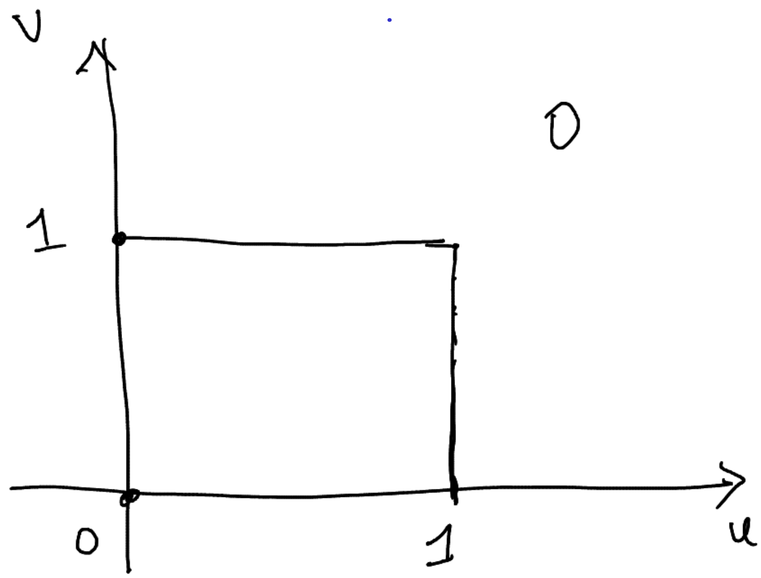
$P(X \leq a, Y \leq b) =$

If such a joint pdf exists then X and Y are

called jointly continuous.

Ex: X, Y have joint pdf

$$f_{XY}(u,v) = \begin{cases} \frac{3}{2}(uv^2+v) & 0 \le u \le 1 \\ & 0 \le v \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



1) Verify that its a probab. density function.

2) Find the marginal density $f_X(x)$ (pdf of X).

There are two things to verify:

1) $f_{X,Y}(a,b) \geqslant 0 \qquad a \in \mathbb{R}$
$b \in \mathbb{R}.$

2) Probabilities add up to 1.

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty} f_{X,Y}(a,b)\,da\,db = 1$$

This is nearly identical to the single variable case when we were discussing pdfs.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\,du\,dv$$

$$= \int_0^1\int_0^1 \frac{3}{2}\left(uv^2+v\right)du\,dv\,.$$

$$= \int_0^1 \frac{3}{2}\left(\left[\frac{u^2v^2}{2}\right]_0^1 + \left[uv\right]_0^1\right)dv$$

$$= \int_0^1 \frac{3}{4}v^2 + \frac{3}{2}v \quad dv$$

$$= \frac{1}{4} + \frac{3}{4} = 1 \quad !$$

Marginal pdfs

Recall how we found marginal distributions in the discrete case: we summed over rows (or columns).

A similar logic applies:

$$f_X(u) =$$

$$\therefore \int_0^1 \frac{3}{2}\left(uv^2+v\right)dv$$

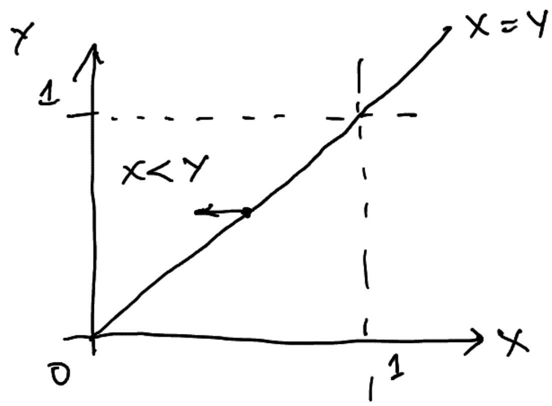
$$= \left(\frac{\cancel{3}}{2}\frac{uv^3}{\cancel{3}}+\frac{3v^2}{2\cdot 2}\right)\Bigg|_0^1$$

$$f_X(u) = \frac{u}{2}+\frac{3}{4} \qquad 0 \le u \le 1.$$

The joint density function contains information about both X and Y. So obviously you can get information about X alone.

Integrating over y ≡ "Forgetting info about Y"

Next, lets find the probability $\mathbb{P}(X<Y)$. Draw a picture first.



This looks pretty symmetric

POLL

$\mathbb{P}(X<Y) = 1/2$	$\mathbb{P}(X<Y) \neq 1/2$

It depends on the function

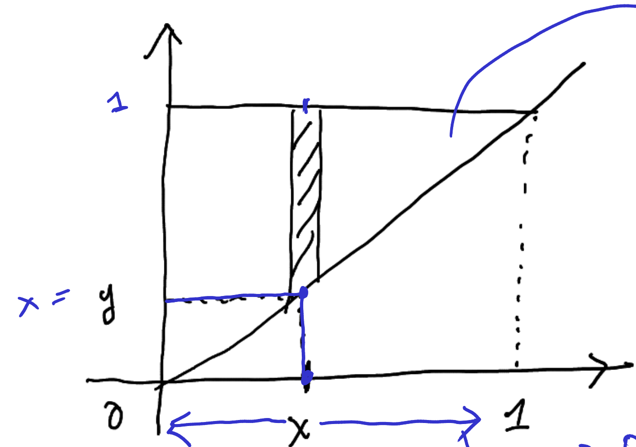
$$f_{XY}(x,y) = \frac{3}{2}(xy^2+y)$$

↑ increases in x and increases in y.

So it gives higher weight to larger x values and larger y values. My guess is that it gives more weight to larger y values, so $\mathbb{P}(X<Y) > \frac{1}{2}$.

Lets see what <u>really</u> happens.

Careful DOUBLE INTEGRAL



Integrate $f_{XY}(x,y)$ over this region

outer integral.

First variable : x , has range 0 to 1.

2nd variable: y goes from x to 1

$$\int_0^1 \int_x^1 f_{X,Y}(x,y)\, dy\, dx$$

$$\int xy^2 + y\, dy = \frac{xy^3}{3} + \frac{y^2}{2}$$

$$= \int_0^1 \int_x^1 \frac{3}{2}(xy^2+y)\, dy\, dx$$

$$= \frac{3}{2} \int_0^1 \left(\frac{xy^3}{3} + \frac{y^2}{2} \right)\Big]_x^1 dx$$

$$\int \frac{x(1-x^3)}{3} + \frac{1-x^2}{2}\,dx = \frac{x^2}{2\cdot 3} - \frac{x^5}{3\cdot 5} + \frac{x}{2} - \frac{x^3}{2\cdot 3}$$

$$= \frac{3}{2}\int_0^1 \frac{x(1^3-x^3)}{3} + \frac{(1^2-x^2)}{2}\,dx$$

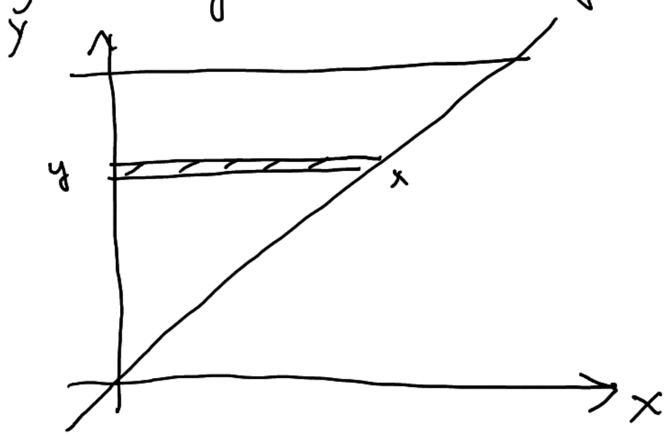
$$= \frac{3}{2}\left[\frac{x^2}{6} - \frac{x^5}{15} + \frac{x}{2} - \frac{x^3}{6}\right]_0^1$$

$$= \frac{3}{2}\left[\cancel{\frac{1}{6}} - \frac{1}{15} + \frac{1}{2} - \cancel{\frac{1}{6}}\right]$$

$$= \frac{3}{2}\left[\frac{15-2}{15\times 2}\right]$$

$$= \frac{\overset{1}{\cancel{3}}}{2}\cdot\frac{13}{\overset{5}{\cancel{15}}\cdot 2} = \frac{13}{20}$$

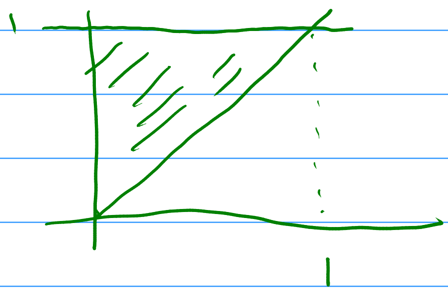
You can also integrate the "other way"
ie integrate over x first and then y.



POLL

Integrate over the region $X < Y$ by integrating over x first.

$$f_{XY}(u,v) = \frac{3}{2}(uv^2 + v) \quad \text{in } [0,1]^2$$



Q&A Find $P(X<Y)$ by integrating over x first

Next find $E[X^2Y]$

$$E[X^2Y] = \int_0^1\int_0^1 x^2y\left(\frac{3}{2}(xy^2+y)\right)dx\,dy$$

$$= \frac{3}{2}\iint (x^3y^3 + x^2y^2)\,dx\,dy$$

Alternate way of finding the pdf of X

(the marginal density $f_X(u)$)

$\mathbb{P}(X \leq u) = F_X(u)$. We will differentiate to get $f_X(u) = \frac{d}{du} F_X(u)$.

$$\mathbb{P}(X \leq u) = \int_0^1 \int_0^4 \frac{3}{2}(xy^2+y)\, dx\, dy$$

$$= \int_0^4 \frac{3}{2}\left(\frac{x}{3} + \frac{1}{2}\right) dx.$$

$$\frac{d}{du}\mathbb{P}(X \leq u) = f_X(u) = \frac{3}{2}\left(\frac{u}{3} + \frac{1}{2}\right)$$

$$\therefore \quad f_X(u) = \int_0^1 f_{X,Y}(u,y)\, dy$$

"Integrate over y to forget it"

Generalities

Let us abstract:

If you have $X_1, \ldots X_n$, and joint pdf $f(x_1, \ldots x_n)$

then for any (nice) set B.

(Ex: B was the triangle)

$$\mathbb{P}\left((X_1, X_2, \cdots X_n) \in B\right)$$

$$= \int \cdots \int_B f(u_1, \cdots u_n)\, du_1 \cdots du_n$$

(Integral over the set B).

Marginal density of X_1

$$f_{X_1}(u_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, u_2, \cdots u_n)\, du_2 \cdots du_n$$

Given $g(u_1 \cdots u_n)$,

$g(X_1, \ldots X_n)$ is a random variable

$$\mathbb{E}[g(X_1, \ldots, X_n)]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(u_1, \ldots u_n)\, du_1 \ldots du_n$$

I usually skip 6.17 which simply has more complicated integration.

<u>Uniform distribution</u> in higher dim

D = domain.

Let $X \sim$ Uniform (D).

(Recall $X \sim$ Uniform $[0,1]$)

Reasonable definition for pdf:

$$f_{XY}(u,v) = \begin{cases} \dfrac{1}{\text{Area of } D} & \text{when } (u,v) \in D \\ 0 & \text{otherwise.} \end{cases}$$

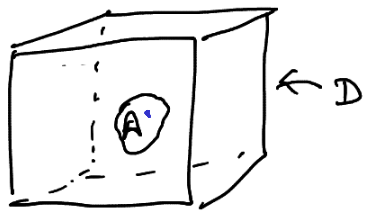
Then $\mathbb{P}(X$ lies in the set $A)$

$$= \mathbb{P}(X \in A) = \iint_A \frac{1}{\text{Area of } D}\, dx\, dy$$

= (by properties of the double integral)

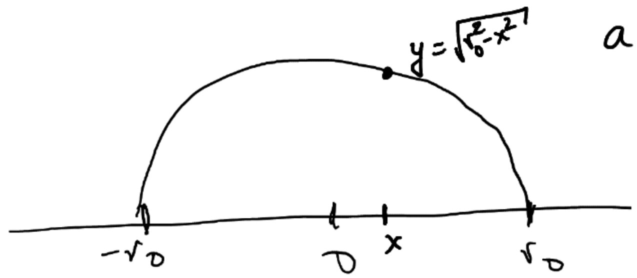
$$= \frac{\text{Area of } A}{\text{Area of } D.} \quad \left(\text{as expected}\right)$$

You can also do this in 3 or higher D:



$$P(X \in A) = \frac{\text{Volume } A}{\text{Volume } D}$$

Ex: Let $X \sim \text{Uniform}(D)$ where D is a half disk.



Find joint pdf $f_{XY}(u,v)$

and Marginal $f_X(u)$.

$$f_{XY}(u,v) = \begin{cases} \frac{1}{\pi r_0^2/2} & (u,v) \in D \\ 0 & \text{otherwise.} \end{cases}$$

What is $f_X(u)$.

$$= \begin{cases} \cdots & -r_0 < u < r_0 \\ 0 & \text{otherwise} \end{cases}$$

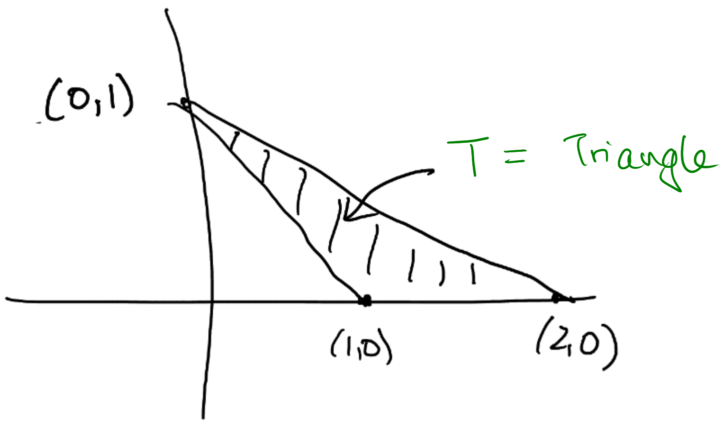
$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\, dv$$

$$= \int_{0}^{\sqrt{r_0^2 - u^2}} \frac{1}{\pi r_0^2/2}\, dv$$

since $x^2 + y^2 = r_0^2$ on circle boundary.

$$f_X(u) = \begin{cases} \dfrac{\sqrt{r_0^2 - u^2}}{\pi r_0^2/2} & -r_0 < u < r_0 \\ 0 & \text{otherwise} \end{cases}$$

Ex triangle: $X \sim \text{Uniform}(\text{triangle})$



$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area of } T} & (x,y) \in G \\ 0 & (x,y) \notin G \end{cases}$$

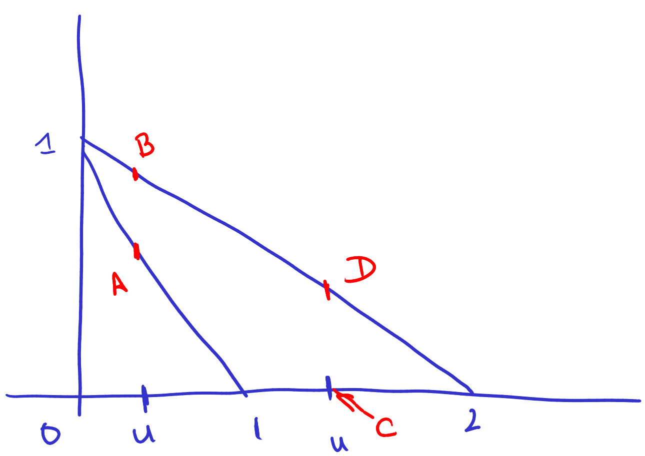
Find marginal density of X and Y.

Marginal of $f_X(u)$

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$$

range (X) =

Equations of the lines



$A = (u, 1-u)$ $\quad$ $C =$

$B = \left(u, 1-\frac{u}{2}\right)$ $\quad$ $D =$

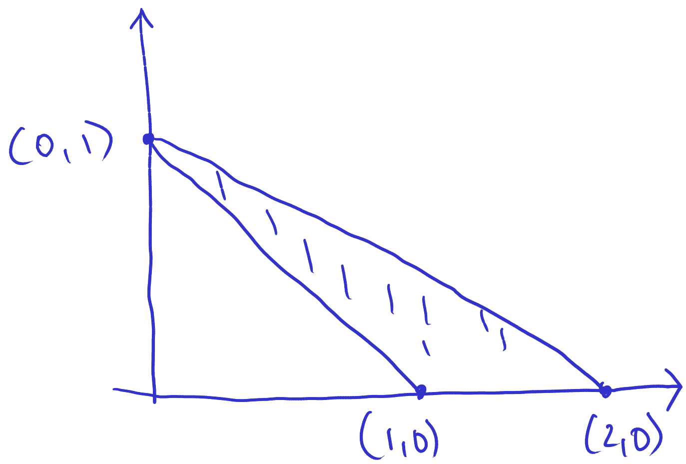
$$f_X(u) = \int_{1-u}^{1-u/2} 2\, dv = 2\left(1-\frac{u}{2} - (1-u)\right) = \frac{2u}{2} = u \qquad 0 \le u \le 1$$

$$f_X(u) = \int_{0}^{1-u/2} 2\, dv = 2\left(1-\frac{u}{2}\right) = 2-u \qquad 1 \le u \le 2$$

POLL

The marginal density for Y is: $f_Y(t)$

Q and A style: $(f(t), \quad a \leq t \leq b)$



range of Y.

$$f_Y(t) = \int_{-\infty}^{\infty} f_{XY}(u,t)\,du$$